

The following is a direct copy of this published paper, but with numerous many printer's errors corrected and with a few additional remarks.

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NONSTANDARD CONSEQUENCE OPERATORS

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(Dedicated to Professor K. Iséki)

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1. Introduction

In 1963, Abraham Robinson applied his newly discovered nonstandard analysis to formal first-order languages and developed a nonstandard logic [11] relative to the “truth” concept and structures. Since that time not a great deal of fundamental research has been attempted in this specific area with one notable exception [3]. However, when results from this discipline are utilized they have yielded some highly significant and important developments such as those obtained by Henson [4].

The major purpose for this present investigation is to institute formally a more general study than previously pursued. In particular, we study nonstandard logics relative to consequence operators [2] [6] [12] [13] defined on a nonstandard language. Since the languages considered are not obtained by the usual constructive methods, then this will necessitate the construction of an entirely new foundation distinctly different from Robinson's basic embedding techniques. Some very basic results of this research were very briefly announced in a previous report [6].

In order to remove ambiguity from the definition of the “finite” consequence operator, the definition of “finite” is the ordinary definition in that the empty set is finite and any nonempty set A is finite if and only if there exists a bijection $f: A \rightarrow [1, n]$, where $[1, n] = \{x \mid n \in \mathbb{N}, 1 \leq x \leq n\}$ (\mathbb{N} is the set of natural numbers with zero). Unless otherwise stated, all sets B that are infinite will also be assumed to be Dedekind-infinite. This occurs when a set B is denumerable, since B inherits a well-ordering from \mathbb{N} , or B is well-ordered [2, p. 248], or the Axiom of Choice is assumed. We note that within mathematics one is always allowed to make a finite choice from finitely many nonempty sets, among others [9, p. 1].

In 2, we give the basic definitions, notations and certain standard results are obtained that indicate the unusual behavior of the algebra of all consequence operators defined on a set. In 4, some standard properties relative to subalgebras and chains in the set of all consequence operators are investigated. Finally, the entire last section is devoted to the foundations of the theory of nonstandard consequence operators defined on a nonstandard language.

2. Basic concepts

Our notations and definitions for the standard theory of consequence operators are taken from references [2][6][12][13], and we now recall the most pertinent of these. Let L be any nonempty set

that is often called a *language*, $\mathcal{P}(L)$ denote the power set of L and for any set X let $F(X)$ denote the finite power set of X (i.e. the set of all finite subsets of X .)

DEFINITION 2.1 A mapping $C: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ is a consequence operator (or closure operator) if for each $X, Y \in \mathcal{P}(L)$

- (i) $X \subset C(X) = C(C(X)) \subset L$ and if
- (ii) $X \subset Y$, then $C(X) \subset C(Y)$.

A consequence operator C defined on L is said to be *finite* (*finitary*, or *algebraic*) if it satisfies

- (iii) $C(X) = \cup\{C(A) \mid A \in F(X)\}$.

REMARK 2.2 The above axioms (i) (ii) (iii) are not independent. Indeed, (i)(iii) imply (ii).

Throughout this entire article the symbol “ C ” with or without subscripts or with or without symbols juxtapositioned to the right will always denote a consequence operator. The only other symbols that will denote consequence operators are “ I ” and “ U ”. The symbol \mathcal{C} [resp. \mathcal{C}_f] denotes the set of all consequence operators [resp. finite consequence operators] defined on $\mathcal{P}(L)$.

DEFINITION 2.3. (i) Let I denote the identity map defined on $\mathcal{P}(L)$.

(ii) Let $U: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ be defined as follows: for each $X \in \mathcal{P}(L)$, $U(X) = L$.

(iii) For each $C_1, C_2 \in \mathcal{C}$, define $C_1 \leq C_2$ iff $C_1(X) \subset C_2(X)$ for each $X \in \mathcal{P}(L)$. (Note that \leq is obviously a partial order defined on \mathcal{C} .)

(iv) For each $C_1, C_2 \in \mathcal{C}$, define $C_1 \vee C_2: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ as follows: for each $X \in \mathcal{P}(L)$, $(C_1 \vee C_2)(X) = C_1(X) \cup C_2(X)$.

(v) For each $C_1, C_2 \in \mathcal{C}$, define $C_1 \wedge C_2: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ as follows: for each $X \in \mathcal{P}(L)$, $(C_1 \wedge C_2)(X) = C_1(X) \cap C_2(X)$.

(vi) For each $C_1, C_2 \in \mathcal{C}$ define $C_1 \vee_w C_2: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ as follows: for each $X \in \mathcal{P}(L)$, $(C_1 \vee_w C_2)(X) = \cap\{Y \mid X \subset Y \subset L \text{ and } Y = C_1(Y) = C_2(Y)\}$.

Prior to defining certain special consequence operators notice that $I, U \in \mathcal{C}_f$ and that I [resp. U] is a lower [resp. upper] unit for the algebras $\langle \mathcal{C}, \leq \rangle$ and $\langle \mathcal{C}_f, \leq \rangle$.

DEFINITION 2.4. Consider any $X, Y \in \mathcal{P}(L)$.

(i) Define $C(X, Y): \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ as follows: let $A \in \mathcal{P}(L)$. If $A \cap Y \neq \emptyset$, then $C(X, Y)(A) = A \cup X$. If $A \cap Y = \emptyset$, then $C(X, Y)(A) = A$.

(ii) Define $C'(X, Y): \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ as follows: let $A \in \mathcal{P}(L)$. If $Y \subset A$, then $C'(X, Y)(A) = A \cup X$. If $Y \not\subset A$, then $C'(X, Y)(A) = A$.

THEOREM 2.5. For each $X, Y \in \mathcal{P}(L)$, $C(X, Y) \in \mathcal{C}_f$ and $C'(X, Y) \in \mathcal{C}$. If $Y \in F(L)$, then $C'(X, Y) \in \mathcal{C}_f$.

PROOF. Let $X, Y, A \in \mathcal{P}(L)$ and consider $C(X, Y)$. If $A \cap Y \neq \emptyset$, then $C(X, Y)(A) = A \cup X \supset A$. If $A \cap Y = \emptyset$, then $C(X, Y)(A) = A$. Hence, for each $A \in \mathcal{P}(L)$, $A \subset C(X, Y)(A)$. Assume $A \cap Y \neq \emptyset$. Then $C(X, Y)(C(X, Y)(A)) = C(X, Y)(A \cup X) = A \cup X = C(X, Y)(A)$, since $(A \cup X) \cap Y \neq \emptyset$. If $A \cap Y = \emptyset$, then $C(X, Y)(C(X, Y)(A)) = C(X, Y)(A)$. Thus, axiom (i) of definition 2.1 holds. Let $A \subset H \subset L$. If $A \cap Y \neq \emptyset$, then $H \cap Y \neq \emptyset$ implies that $C(X, Y)(A) = A \cup X \subset H \cup X = C(X, Y)(H)$. Assume that $A \cap Y = \emptyset$. Then $C(X, Y)(A) = A$. If $H \cap Y \neq \emptyset$, then $A \subset H \cup X = C(X, Y)(H)$. If $H \cap Y = \emptyset$, then $A \subset H = C(X, Y)(H)$. Thus, in all cases, $C(X, Y)(A) \subset C(X, Y)(H)$ and axiom (ii) holds. Let $A \cap Y \neq \emptyset$ and $x \in C(X, Y)(A) = A \cup X$. If $x \in A$, then $C(X, Y)(\{x\}) = \{x\} \cup X$ or $\{x\}$. Hence, in this case, $x \in C(X, Y)(\{x\})$. Suppose that $x \in X$. Then there exists some $y \in A \cap Y$ and $x \in C(X, Y)(\{y\}) = \{y\} \cup X \subset A \cup X$. Consequently, if $A \cap Y \neq \emptyset$ and $x \in C(X, Y)(A)$,

then there is some $F \in F(A)$ such that $x \in C(X, Y)(F)$. Consider the case where $A \cap Y = \emptyset$. If $A = \emptyset$, $C(X, Y)(A) = \emptyset = \bigcup \{C(X, Y)(F) \mid F \in F(\emptyset)\}$. Let $A \neq \emptyset$, then $x \in C(X, Y)(A) = A$ implies that $x \in C(X, Y)(\{x\})$ and $\{x\} \in F(A)$. Hence, in general, if $x \in C(X, Y)(A)$, then $x \in \bigcup \{C(X, Y)(F) \mid F \in F(A)\}$. Since $C(X, Y)(F) \subset C(X, Y)(A)$ for each $F \in F(A)$, then it follows that (iii) holds.

Consider $C'(X, Y)$, let $A \in \mathcal{P}(L)$ and assume that $Y \subset A$. Then $C'(X, Y)(A) = A \cup X \supset A$. Moreover, $C'(X, Y)(C'(X, Y)(A)) = C'(X, Y)(A \cup X) = A \cup X = C'(X, Y)(A)$. If $Y \not\subset A$, then $C'(X, Y)(A) = A$ and $C'(X, Y)(C'(X, Y)(A)) = C'(X, Y)(A)$. Thus axiom (i) holds. The fact that if $A \subset H \subset L$, then $C'(X, Y)(A) \subset C'(X, Y)(H)$ follows easily and (ii) holds.

Assume that $Y \in F(L)$, $Y \subset A$ and $x \in C'(X, Y)(A) = A \cup X$. If $x \in X$, then $x \in C'(X, Y)(Y) = Y \cup X \subset A \cup X$. If $x \in A$, then $x \in C'(X, Y)(Y \cup \{x\}) = Y \cup \{x\} \cup X \subset A \cup X$. But $Y \cup \{x\} \in F(A)$. Hence, in this case, $x \in C'(X, Y)(F)$, where $F \in F(A)$. Finally, let $Y \not\subset A$. If $A = \emptyset$, $C'(X, Y)(A) = \emptyset = \bigcup \{C'(X, Y)(F) \mid F \in F(\emptyset)\}$. Assume $A \neq \emptyset$ and $x \in C'(X, Y)(A) = A$. Then $x \in A$. If $A \subset Y$, $A \neq Y$, then $x \in C'(X, Y)(\{x\}) = \{x\}$. Otherwise, $A \not\subset Y$ and there exists some $z \in A$ such that $z \notin Y$. In which case, $Y \not\subset \{x, z\}$ and, hence, $x \in C'(X, Y)(\{x, z\}) = \{x, z\} \in F(A)$. Therefore, $C(X, Y)(A) \subset \bigcup \{C'(X, Y)(F) \mid F \in F(A)\}$. This result and axiom (ii) imply that axiom (iii) holds. This completes the proof.

Recall that $C \in \mathcal{C}$ is *axiomless* if $C(\emptyset) = \emptyset$ and axiomatic otherwise [8]. Note that for any $X, Y \in \mathcal{P}(L)$, $C(X, Y)$ is axiomless, and if $X = \emptyset$ or $Y \neq \emptyset$, then $C'(X, Y)$ is axiomless.

LEMMA 2.6. *Let $C \in \mathcal{C}$ be axiomatic. Then there exists some $x \in L$ such that $C(L - \{x\}) = L$.*

PROOF. Assume that there does not exist some $x \in L$ such that $C(L - \{x\}) = L$. Then for each $y \in L$, $L - \{y\} \subset C(L - \{y\}) \subset L$ implies that $L - \{y\} = C(L - \{y\})$ from axiom (i). But axiom (ii) yields that $C(\emptyset) = C(\bigcap \{L - \{y\} \mid y \in L\}) \subset \bigcap \{C(L - \{y\}) \mid y \in L\} = \bigcap \{L - \{y\} \mid y \in L\} = \emptyset$. Thus C would be axiomless and this contradiction completes the proof.

Recall that a member C_1 in the algebra $\langle \mathcal{C}, \leq \rangle$ covers $C_2 \in \mathcal{C}$, if $C_2 < C_1$ and there does not exist some $C_3 \in \mathcal{C}$ such that $C_2 < C_3 < C_1$. A set $\mathcal{B} \subset \mathcal{C}$ *densely* covers $\mathcal{E} \subset \mathcal{C}$ if for each $C \in \mathcal{B}$ there exists some $C_1 \in \mathcal{E}$ such that $C_1 \leq C$. Recall that $C \in \mathcal{C}$ is an *atom* if C covers I and $\langle \mathcal{C}, \leq, I \rangle$ is *atomic* with $\mathcal{E} \subset \mathcal{C}$ the set of atoms if each member of \mathcal{E} covers I and for each $I \neq C \in \mathcal{C}$ there is some $C_1 \in \mathcal{E}$ such that $C_1 \leq C$. Let $\mathcal{E}_0 = \{C'(\{x\}, L - \{x\}) \mid x \in L\}$. Notice that $I \notin \mathcal{E}_0$, since for $x \in L$, $C'(\{x\}, L - \{x\})(L - \{x\}) = L$, and that each member of \mathcal{E}_0 is axiomless if L has more than one member. The next result shows that $\langle \mathcal{C}, \leq \rangle$ is almost atomic.

THEOREM 2.7. *For $\langle \mathcal{C}, \leq \rangle$, the set all axiomatic consequence operators \mathcal{C}_A covers \mathcal{E}_0 and each member of \mathcal{E}_0 is a atom.*

PROOF. We first show that each member of \mathcal{E}_0 is an atom. Let $x \in L$ and assume that there exists some $C_1 \in \mathcal{C}$ such that $C_1 < C'(\{x\}, L - \{x\}) = C'$. Assume that $B \in \mathcal{P}(L)$, $C_1(B) \subset C'(B)$ and $C_1(B) \neq C'(B)$. Suppose that $L - \{x\} \subset B$. Then $C'(B) = L$ and $L - \{x\} \subset B \subset C_1(B) \neq L$. Hence, $L - \{x\} = B = C_1(B)$. Now suppose that $L - \{x\} \not\subset B \neq L$. Then $B \subset C_1(B) \subset C'(B) = B$. However, this contradicts $C_1(B) \neq C'(B)$. Thus for any $B \in \mathcal{P}(L)$ such that $C_1(B) \neq C'(B)$, it follows that $L - \{x\} \subset B$ and $C_1(B) = B$ and there is only one such set with these properties, the set is $L - \{x\}$ since $C_1(L) = C'(L) = L$. Therefore, it must follow that $L - \{x\} = B = C_1(B) = I(B)$ and $C'(L) = L$ in order that $C_1(B) \subset C'(B)$ and $C_1(B) \neq C'(B)$. Further, in general, $I(B) \subset C'(B) = L$ and $I(B) \neq C'(B)$. Now if $A \in \mathcal{P}(L)$ and $B \neq A \neq L$, then $L - \{x\} \not\subset A$ implies that $C_1(A) = C'(A) = A$. Finally, if $A = L$, then $C_1(L) = C'(L) = L = I(L)$. Consequently, $C_1 = I$, $C_1 < C'$ and there is no $C \in \mathcal{C}$ such that $C_1 < C < C'$. Hence, C' is an atom.

We now easily show that \mathcal{C}_A densely covers \mathcal{E}_0 . Let $C \in \mathcal{C}_A$. Then from Lemma 2.6, there exists some $x \in L$ such that $C(L - \{x\}) = L$. Then letting $C'(\{x\}, L - \{x\}) = C'$, it follows that $C'(L - \{x\}) = L = C(L - \{x\})$. If $A = L$, then $C'(A) = C(A)$ and if $A \subset L$, $A \neq L - \{x\}$, $A \neq L$, then $C'(A) = A \subset C(A)$. Hence, $C' \leq C$. This completes the proof.

It is not difficult to show that $\langle \mathcal{C}, \leq \rangle$ [13] and $\langle \mathcal{C}_f, \leq \rangle$ are both closed under \wedge , where \wedge is (v) of definition 2.3, which, obviously, would be the *meet* operation associated with \leq . However, as will be shown by a simple example, it is rare that these algebras are closed under \vee , which if either is so closed, then \vee would be the *join* operation. Wójcicki was the first to recognize that not only are these algebras closed under \vee_w , and \vee_w is the *join* operation, but $\langle \mathcal{C}, \wedge, \vee_w, I, U \rangle$ is also complete [13, p. 276]. Unfortunately, $\langle \mathcal{C}_f, \wedge, \vee_w, I, U \rangle$ is not complete [2, p. 180]. For a simple proof that $\langle \mathcal{C}, \wedge, I, U \rangle$ is meet-complete (and thus complete) see [13, p. 276]. Using the fact that $\langle \mathcal{C}, \leq \rangle$ is a meet semi-lattice, it follows easily that $\langle \mathcal{C}_f, \leq \rangle$ is also a meet semi-lattice. We need only show that, for $\langle \mathcal{C}_f, \leq \rangle$, \wedge satisfies axiom (iii). Let $C_1, C_2 \in \mathcal{C}_f$. Then for each $A \in \mathcal{P}(L)$, $(C_1 \wedge C_2)(A) = C_1(A) \cap C_2(A) = (\bigcup \{C_1(X) \mid X \in F(A)\}) \cap (\bigcup \{C_2(X) \mid X \in F(A)\}) = \bigcup \{C_1(X) \cap C_2(X) \mid X \in F(A)\} = \bigcup \{(C_1 \wedge C_2)(X) \mid X \in F(A)\}$ and (iii) holds.

EXAMPLE 2.8. Certain subsets of $\langle \mathcal{C}, \leq \rangle$ and $\langle \mathcal{C}_f, \leq \rangle$ may be closed under \vee , but, in general, there are members such that the \vee operator does not yield a consequence operator. Let L have 3 or more members. Define $S: \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ by letting $\emptyset \neq M \subset L$, $|L - M| \geq 2$. Let $S(\emptyset) = M$ and $b \in L - M$. For each $A \in \mathcal{P}(L)$, if $b \in A$, then let $S(A) = L$; if $b \notin A$, let $S(A) = M \cup A$. It follows easily that $S \in \mathcal{C}_A \cap \mathcal{C}_f$. Consider $C'(\{b\}, \emptyset) \in \mathcal{C}_f$. Then $(C'(\{b\}, \emptyset) \vee S)(C'(\{b\}, \emptyset) \vee S)(\emptyset) = (C'(\{b\}, \emptyset) \vee S)(C'(\{b\}, \emptyset)(\emptyset) \cup S(\emptyset)) = (C'(\{b\}, \emptyset) \vee S)(\{b\} \cup M) = (C'(\{b\}, \emptyset)(\{b\} \cup M) \cup S(\{b\} \cup M)) = \{b\} \cup M \cup L = L \neq (C'(\{b\}, \emptyset) \vee S)(\emptyset) = \{b\} \cup M$. Thus $C'(\{b\}, \emptyset) \vee S$ is not a consequence operator.

Observe that if L is a standard formal propositional or predicate language and S' the propositional or predicate consequence operator respectively, then even though S' is not the same operator as defined in example 2.8 the presence of the formula $b = P \wedge (\neg P)$ will also yield that $C'(\{b\}, \emptyset) \vee S' \notin \mathcal{C}$. Simply substitute S' for S with this formula as the b and notice that $L - S'(\emptyset)$ is denumerable, $b \in (L - S'(\emptyset))$, and $S'(\{b\}) \cup S'(\emptyset) = L$.

3. Subalgebras

Subalgebras of $\langle \mathcal{C}, \wedge, \vee_w, I, U \rangle$ and $\langle \mathcal{C}_f, \wedge, \vee_w, I, U \rangle$ have been studied to a certain extent and appear to be the most appropriate area for further investigation. It is known that there are sublattices of $\langle \mathcal{C}, \wedge, \vee_w, I, U \rangle$ that are atomic and coatomic [2, p. 179]. We first show that there are complete and distributive sublattices of $\langle \mathcal{C}_f, \wedge, \vee_w, I, U \rangle$, where $\vee = \vee_w$. Moreover, such sublattices need not be atomic.

THEOREM 3.1. *For each $B \in \mathcal{P}(L)$, let $\mathcal{C}(B) = \{C(X, B) \mid X \in \mathcal{P}(L)\}$. Then $\langle \mathcal{C}(B), \wedge, \vee, I, C(L, B) \rangle$ is a complete and distributive sublattice of $\langle \mathcal{C}_f, \wedge, \vee_w, I, U \rangle$. If there exists nonempty $A, B \in \mathcal{P}(L)$, such that $A \neq L$, $B \subset A$, and $B \neq A$, then $\langle \mathcal{C}(B), \wedge, \vee, I, C(L, B) \rangle$ is not a chain.*

PROOF. Let $A, B, X, Y \in \mathcal{P}(L)$. Note that $C(\emptyset, B) = I$. Let \mathcal{H} be any nonempty subset of $\mathcal{C}(B)$. Then there exists some nonempty $\mathcal{A} \subset \mathcal{P}(L)$ such that $\mathcal{H} = \{C(A, B) \mid A \in \mathcal{A}\}$. We first show that $\inf(\mathcal{H}) = C(\bigcap \mathcal{A}, B)$. If $B \cap X \neq \emptyset$, then $C(\bigcap \mathcal{A}, B)(X) = (\bigcap \mathcal{A}) \cup X \subset C(A, B)(X) = A \cup X$ for each $A \in \mathcal{A}$. If $B \cap X = \emptyset$, then $C(\bigcap \mathcal{A}, B)(X) = X = C(A, B)(X)$ for each $A \in \mathcal{A}$. Thus $C(\bigcap \mathcal{A}, B) \leq C(A, B)$ for each $A \in \mathcal{A}$. Hence, $C(\bigcap \mathcal{A}, B)$ is a lower bound for \mathcal{H} . Let $C(Y, B)$ be a lower bound for \mathcal{H} . If $B \cap X \neq \emptyset$, then $Y \cup X \subset A \cup X$ for all $A \in \mathcal{A}$ yields that $Y \cup X \subset \bigcap \{(A \cup X) \mid$

$A \in \mathcal{A}\} = (\cap \mathcal{A}) \cup X = C(\cap \mathcal{A}, B)(X)$. If $B \cap X = \emptyset$, then $C(Y, B)(X) = X = C(A, B)(X) = C(\cap \mathcal{A}, B)(X)$ where $A \in \mathcal{A}$. Thus $C(Y, B) \leq C(\cap \mathcal{A}, B)$ implies that $\inf(\mathcal{H}) = C(\cap \mathcal{A}, B)$. We next show that $\sup(\mathcal{H}) = C(\cup \mathcal{A}, B)$. But, first, we show that $\vee = \vee_w$. Let $A_1 \in \mathcal{P}(L)$ and assume that $B \cap X = \emptyset$. Then $(C(A, B) \vee C(A_1, B))(X) = X$. Now from the definition of \vee_w [13, p. 176], $(C(A, B) \vee_w C(A_1, B))(X) = \bigcap \{Y \mid X \subset Y = C(A, B)(Y) = C(A_1, B)(Y)\} = X$ since $C(A, B)(X) = C(A_1, B)(X) = X$ in this case. Now let $B \cap X \neq \emptyset$. Then $(C(A, B) \vee C(A_1, B))(X) = A \cup A_1 \cup X$. If $X \subset C(A, B)(Y) = Y = C(A_1, B)(Y)$, then $Y \cap B \neq \emptyset$ yields that $A \cup Y = Y = A_1 \cup Y$. Hence, letting $Y_1 = A \cup A_1 \cup X \subset Y$, then $C(A, B)(Y_1) = Y_1 = C(A_1, B)(Y_1)$. Thus $(C(A, B) \vee_w C(A_1, B))(X) = \bigcap \{Y \mid X \subset Y = C(A, B)(Y) = C(A_1, B)(Y)\} = A \cup A_1 \cup X$. Consequently, $\vee = \vee_w$. It now follows, in like manner, that $\sup(\mathcal{H}) = C(\cup \mathcal{A}, B)$. The fact that this complete sublattice is distributive follows from the fact that \cup and \cap are distributive.

For the final part, assume that nonempty $A, B \in \mathcal{P}(L)$, $A \neq L$, $B \subset A$ and $B \neq A$. There is some $x \in L - A$. Let $D = \{x\}$. Then $C(A, B)(B) = A \cup B = A$ and $C(D, B)(B) = B \cup D$. However, $A \not\subset B \cup D$ and $B \cup D \not\subset A$ imply that $C(A, B)$ and $C(D, B)$ are not comparable. Thus in this case, $\langle \mathcal{C}(B), \wedge, \vee, I, C(L, B) \rangle$ is not a chain and the proof is complete.

Using the collection of axiomless consequence operators defined in theorem 3.1, we show, in general, that the algebras \mathcal{C} and \mathcal{C}_f do not have the descending chain condition.

EXAMPLE 3.2. For infinite L , we show that for a specific B , $\langle \mathcal{C}(B), \wedge, \vee, I, C(L, B) \rangle$ contains a $\langle \mathcal{C}_f, \leq \rangle$ chain that does not satisfy the descending chain condition. Hence, $\langle \mathcal{C}_f, \leq \rangle$ and $\langle \mathcal{C}, \leq \rangle$ would not satisfy the descending chain condition. There exists an injection $F: \mathbb{N} \rightarrow L$. Let $f(0) = x_0$ and $B = \{x_0\}$. For each $n \in \mathbb{N}$, $n \geq 1$, let $F_n = L - f[[1, n]]$ and $C_n = C(F_n, B)$. Notice that no $F_n = \emptyset$. Let $\mathcal{H} = \{C_n \mid n \geq 1\}$. Let distinct $C_k, C_m \in \mathcal{H}$. Then either $k < m$ or $m < k$. Assume that $k < m$. In general, let $C_n \in \mathcal{H}$, $X \in \mathcal{P}(L)$. If $x_0 \in X$, $C_n(X) = F_n \cup X$. If $x_0 \notin X$, $C_n(X) = X$. Thus, for any $n < m$, $\emptyset \neq F_m \subsetneq F_n$ implies that $C_m \not\leq C_k$. In like manner, for $m < k$. Hence, any two members of \mathcal{H} are comparable with respect to $\langle \mathcal{C}_f, \leq \rangle$. Also for each C_n , $C_n(\{x_0, x_1\}) = F_n \cup \{x_0\} = F_n \neq \{x_0, x_1\}$. Therefore, no $C_n = I$. This yields that for $n \in \mathbb{N}$, $n \geq 1$, $I \neq C_{n+1} \not\leq C_n$ and this completes the example. Notice that by taking $A = \{x_0, x_1\}$, $\mathcal{C}(B)$ is not a chain.

Since $\langle \mathcal{C}, \wedge, \vee_w, I, U \rangle$ contains distributive and complete sublattices that are not chains, where $\vee = \vee_w$, then a natural question to ask is whether or not such sublattices can have any other Boolean type structures?

THEOREM 3.3. Let $C, C_1 \in \mathcal{C}$ and $I \not\leq C \not\leq C_1$. Then $C' \in \mathcal{C}$ is a complement relative to C and with respect to \vee iff for each $A \in \mathcal{P}(L)$, $C'(A) = (C_1(A) - C(A)) \cup A$.

PROOF. Let $C, C_1 \in \mathcal{C}$, and $I \not\leq C \not\leq C_1$. The C' is a relative complement for C iff $C \vee C' = C_1$ and $C \wedge C' = I$ iff for each $A \in \mathcal{P}(L)$, (1) $C(A) \cup C'(A) = C_1(A)$ and (2) $C(A) \cap C'(A) = A$. For the necessity, assume (1) and (2) and let $y \in (C_1(A) - C(A)) \cup A$. If $y \in A$, then $y \in C'(A)$ by (2). If $y \notin A$, then $y \in C_1(A) - C(A)$ implies that $y \in C_1(A)$ and $y \notin C(A)$; which further implies that $y \in C'(A)$ from (1). On the other hand, if $y \in C'(A)$ and $y \notin A$, then by (1) and (2), $y \in C_1(A) - C(A)$ implies that $y \in (C_1(A) - C(A)) \cup A$. Obviously, if $y \in A$, then $y \in (C_1(A) - C(A)) \cup A$. Thus if (1) and (2) holds, then $C'(A) = (C_1(A) - C(A)) \cup A$ for each $A \in \mathcal{P}(L)$.

For the sufficiency, let $C'(A) = (C_1(A) - C(A)) \cup A$ for each $A \in \mathcal{P}(L)$. Then $C(A) \cup C'(A) = C_1(A) \cup A = C_1(A)$ and $C'(A) \cap C(A) = ((C_1(A) - C(A)) \cup A) \cap C(A) = \emptyset \cup A = A$ and this completes the proof.

EXAMPLE 3.4. We show that \mathcal{C} is not closed under composition of maps even though such a composition is always defined. Let S be the consequence operator defined in example 2.8.

Then $\emptyset \neq S(\emptyset) = M \neq L$, $b \in L - M$ and there exists some $a \in L - M$ such that $a \neq b$. Hence, denoting composition by juxtaposition, it follows that $(C'(\{b\}, \emptyset)S): \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ and $(C'(\{b\}, \emptyset)S)(M) = C'(\{b\}, \emptyset)(M) = M \cup \{b\}$. However, $(C'(\{b\}, \emptyset)S)(M \cup \{b\}) = C'(\{b\}, \emptyset)(L) = L$. Therefore, $(C'(\{b\}, \emptyset)S)(C'(\{b\}, \emptyset)S)(M) = L$. It follows that this composition is not a consequence operator.

Even though example 3.4 shows that \mathcal{C} is not closed under composition the next proposition gives a strong characterization for chains in terms of composition.

THEOREM 3.5. *Let $\mathcal{A} \subset \mathcal{C}$. Then \mathcal{A} is a chain in $\langle \mathcal{C}, \leq \rangle$ iff for each $C, C' \in \mathcal{A}$ either the composition $C'C = C'$ or $CC' = C$.*

PROOF. Let $C, C' \in \mathcal{A}$ and assume that \mathcal{A} is a chain in $\langle \mathcal{C}, \leq \rangle$. Suppose that $C \leq C'$. Then for each $B \in \mathcal{P}(L)$, $B \subset C(B) \subset C'(B)$ implies that $C'(B) \subset C'C(B) \subset C'(C'(B)) = C'(B)$. Hence, $C'C = C'$. In like manner, if $C' \leq C$, then $CC' = C$.

Conversely, let $C, C' \in \mathcal{A}$ and $C'C = C'$. Then for each $B \in \mathcal{P}(L)$, $C(B) \subset C'(C(B)) = (C'C)(B) = C'(B)$. Thus $C \leq C'$. In like manner, if $CC' = C$, then $C' \leq C$ and this completes the proof.

In the next section, our attention is often restricted to chains in $\langle \mathcal{C}_f, \leq \rangle$. We first embed $\langle \mathcal{C}, \leq \rangle$ into a non-standard structure and investigate nonstandard bounds for various chains.

4. Nonstandard Consequence Operators

Let \mathcal{A} be a nonempty finite set of symbols. It is often convenient to assume that \mathcal{A} contains a symbol that represents a blank space. As usual any nonempty finite string of symbols from \mathcal{A} , with repetitions, is called a *word* [10, p.222]. A word is also said to be an (intuitive) *readable sentence* [5, p. 1]. We let W be the intuitive set of all words created from the *alphabet* \mathcal{A} . Note that in distinction to the usual approach, W does not contain a symbol for the empty word.

We accept the concept delineated by Markov [4], the so-called “abstraction of identity,” and say that $w_1, w_2 \in W$ are “equal” if they are composed of the same symbols written in the same intuitive order (left to right). The *join* or juxtaposition operation between $w_1, w_2 \in W$ is the concept that yields the string w_1w_2 or w_2w_1 . Thus W is closed under join. Notice that we may consider a denumerable formal language as a subset of W . (By adjoining a new symbol not in \mathcal{A} and defining it as the unit, W becomes a free monoid generated by the set $\mathcal{A} \cup \{\text{new symbol}\}$.)

Since W is denumerable, then there exists an injection $i: W \rightarrow \mathbb{N}$. Obviously, if we are working with a formal language that is a subset of W , then we may require i restricted to a formal language to be a Gödel numbering. Due to the join operation, a fixed member of W that contains two or more distinct symbols can be represented by various *subwords* that are joined together to yield the given fixed word. The word “mathematics” is generated by the join of $w_1 = \text{math}$, $w_2 = \text{e}$, $w_4 = \text{mat}$, $w_4 = \text{ics}$. This word can also be formed by joining together 11 not necessarily distinct members of W .

Let $i[W] = T$ and for each $n \in \mathbb{N}$, let $T^n = T^{[0,n]}$ denote the set of all mappings from $[0, n]$ into T . Each element of T^n is called a *partial sequence*, even though this definition is a slight restriction of the usual one that appears in the literature. Let $f \in T^n, n > 0$. Then the *order induced by f* is the simple inverse order determined by f applied to the simple order on $[0, n]$. Formally, for each $f(j), f(k) \in f[[0, n]]$, define $f(k) \leq_f f(j)$ iff $j \leq k$, where \leq is the simple order for \mathbb{N} restricted to $[0, n]$. In general, we will not use this notation \leq_f but rather we will indicate this (finite) order

in the usual acceptable manner by writing the symbols $f(n), f(n-1), \dots, f(0)$ from left to right . Thus we symbolically let $f(n) \leq_f f(n-1) \leq_f \dots \leq_f f(0) = f(n)f(n-1) \dots f(0)$.

Let $f \in t^n$. Define $w_f \in W$ as follows: $w_f = (i^{-1}(f(n)))(i^{-1}(f(n-1))) \dots (i^{-1}(f(0)))$, where the operation indicated by juxtaposition is the join. We now define a relation on $P = \cup\{T^n \mid n \in \mathbb{N}\}$ as follows: let $f, g \in P$. Then for $f \in T^n$ and $g \in T^m$, define $f \sim g$ iff $(i^{-1}(f(n))) \dots (i^{-1}(f(0))) = (i^{-1}(g(m))) \dots (i^{-1}(g(0)))$. It is obvious that \sim is an equivalence relation on P . For each $f \in P$, $[f]$ denotes the equivalence class under \sim that contains f . Finally, let $\mathcal{E} = \{[f] \mid f \in P\}$. Observe that for each $[f] \in \mathcal{E}$ there exist $f_0, f_m \in [f]$ such that $f_0 \in T^0$, $f_m \in T^m$ and if there exists some $k \in \mathbb{N}$ such that $0 < k < m$, then there exists some $g_k \in [f]$ such that $g_k \in T^k$ and if $j \in \mathbb{N}$ and $j > m$, then there does not exist $g_j \in T^j$ such that $g_j \in [f]$. If we define the *size* of a word $w \in W$ ($\text{size}(w)$) to be the number of not necessarily distinct symbols counting left to right that appear in W , then the $\text{size}(w) = m + 1$. For each $w \in W$, there is $f_0 \in T^0$ such that $w = i^{-1}(f_0(0))$ and such an $f_m \in [f_0]$ such that $\text{size}(w) = m + 1$. On the other hand, given $f \in P$, then there is a $g_0 \in [f]$ such that $(i^{-1}(g_0(0))) \in W$. Of course, each $g \in [f]$ is interpreted to be the word $(i^{-1}(g(k))) \dots (i^{-1}(g(0)))$.

Each $[f] \in \mathcal{E}$ is said to be a (formal) word or (formal) *readable sentence*. All the intuitive concepts, definitions and results relative to consequence operators defined for $A \in \mathcal{P}(W)$ are now passed to $\mathcal{P}(\mathcal{E})$ by means of the quotient map θ generated by the equivalence relation \sim . In the usual manner, the quotient map is extended to subsets of each $A \in \mathcal{P}(W)$, n -ary relations and the like. For example, let $w \in A \in \mathcal{P}(W)$. Then there exists $f_w \in P$ such that $f_w \in T^0$ and $f_w(0) = i(w)$. Then $\theta(i(w)) = [f_w]$. In order to simplify notation, the images of the extended (θi) composition will often be indicated by bold notation with the exception of customary relation symbols which will be understood relative to the context. For example, if T is a subset of W , then we write $\theta(i[T]) = \mathbf{T}$.

Let \mathcal{N} be a superstructure constructed from the set \mathbb{N} as its set of atoms. Our standard structure is $\mathcal{M} = (\mathcal{N}, \in, =)$. Let $^*\mathcal{M} = (^*\mathcal{N}, \in, =)$ be a nonstandard and elementary extension of \mathcal{M} . Further, $^*\mathcal{M}$ is an enlargement.

For an alphabet \mathcal{A} , there exists $[g] \in ^*\mathcal{E} - \mathcal{E}$ such that there are only finitely many standard members of \mathbb{N} in the range of g and these standard members injectively correspond to alphabet symbols in \mathcal{A} [5, p. 24]. On the other hand, there exist $[g'] \in ^*\mathcal{E} - \mathcal{E}$ such that the range of $[g']$ does not correspond in this manner to elements in \mathcal{A} [5, p. 90].

Let $C \in \mathcal{H}$ map a family of sets \mathcal{B} into \mathcal{B}_0 . If C satisfies either the Tarski axioms (i), (ii) or (i), (iii), or the * -transfer $^*(i)$, $^*(ii)$, or $^*(i)$, $^*(iii)$ of these axioms, then C is called a *subtle consequence operator*. For example, if $C \in \mathcal{C}$, then it is immediate that $^*\mathbf{C}: ^*(\mathcal{P}(\theta(A)) \rightarrow ^*\mathcal{P}(\theta(A)))$ satisfies $^*(i)$ and $^*(ii)$ for the family of all internal subsets of $^*(\theta(A))$. This $^*\mathbf{C}$ is a subtle consequence operator. For any set $A \in \mathcal{N}$, let $^\sigma A = \{^*a \mid a \in A\}$. (In general, this definition does not correspond to that used by other authors.) If for a subtle consequence operator C there does not exist some similarly defined $D \in \mathcal{N}$ such that $C = ^\sigma \mathbf{D}$ or $C = ^*\mathbf{D}$, then C is called a *purely* subtle consequence operator. Let infinite $A \subset \mathcal{E}$ and $B = ^*A - ^\sigma A$. Then the identity $I: \mathcal{P}(B) \rightarrow \mathcal{P}(B)$ is a purely subtle consequence operator.

There are certain technical procedures associated with the σ map that take on a specific significance for consequence operators. Recall that \mathcal{N} is closed under finitely many power set or finite power set iterations. Let $X, Y \in \mathcal{N}$. It is not difficult to show that if $\mathcal{P}: \mathcal{P}(X) \rightarrow Y$, then for each $A \in \mathcal{P}(X)$, $^*(\mathcal{P}(A)) = ^*\mathcal{P}(^*A)$. Moreover, if $F: \mathcal{P}(X) \rightarrow Y$, where F is the finite power set operator, then for each $A \in \mathcal{P}(A)$, $^*(F(A)) = ^*F(^*A)$. If $C \in \mathcal{C}$ and $X \subset \mathcal{E}$, then $\mathbf{C}: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ has the property that for each $A \in \mathcal{P}(X)$, $^*(\mathbf{C}(A)) = ^*\mathbf{C}(^*A)$.

Recall that we identify each ${}^*n \in {}^*\mathbb{N}$ with $n \in \mathbb{N}$ since *n is but a constant sequence with the value n . Utilizing this fact, we have the following straightforward lemma the proof of which is omitted.

LEMMA 4.1. *Let $A \in \mathcal{N}$ and $C \in \mathcal{C}$.*

- (i) $\sigma(F(A)) = F(\sigma A)$ and if $A \in \mathcal{E}$, then $\sigma(F(A)) = F(A)$.
- (ii) $\sigma(\mathbf{C}(B)) = \mathbf{C}(B)$, for each $B \in \mathcal{P}(X)$, $X \in \mathcal{E}$, and
- (iii) ${}^*\mathbf{C}[\{ {}^*A \mid A \in \mathcal{P}(X) \}] = \{ ({}^*A, {}^*B) \mid (A, B) \in \mathbf{C} \} = {}^\sigma\mathbf{C}$. If $F \in F(B)$, then $\sigma(\mathbf{C}(F)) \subset ({}^\sigma\mathbf{C})(\sigma F) = ({}^\sigma\mathbf{C})(F)$. Also, $\sigma(\mathbf{C}(B)) \subset {}^\sigma\mathbf{C}({}^*B)$ and in general $\sigma(\mathbf{C}(B)) \neq ({}^\sigma\mathbf{C})({}^*B)$, $\sigma(\mathbf{C}(F)) \neq ({}^\sigma\mathbf{C})(F)$.
- (iv) $\sigma(\mathbf{C}(B)) = \bigcup \{ \sigma(\mathbf{C}(F)) \mid F \in \sigma(F(B)) \} = \bigcup \{ \sigma(\mathbf{C}(F)) \mid F \in F(\sigma B) \} = \mathbf{C}(B) = \bigcup \{ \mathbf{C}(F) \mid F \in F(B) \}$, where $C \in \mathcal{C}_f$.

Throughout the remainder of this paper, we remove from \mathcal{C} the one and only one inconsistent consequence operator U . Thus notationally we let \mathcal{C} denote the set of all consequence operators defined on infinite $L \subset W$ with the exception of U . Two types of chains will be investigated. Let \mathbf{T} be any chain in $\langle \mathcal{C}, \leq \rangle$ and \mathbf{T}' be any chain with the additional property that for each $C \in \mathbf{T}'$ there exists some $C' \in \mathbf{T}'$ such that $C < C'$.

THEOREM 4.2 *There exists some $C_0 \in {}^*\mathbf{T}$ such that for each $C \in \mathbf{T}$, ${}^*\mathbf{C} \leq C_0$. There exists some $C'_0 \in {}^*\mathbf{T}'$ such that C'_0 is a purely subtle consequence operator and for each $C \in \mathbf{T}'$, ${}^*\mathbf{C} < C'_0$. Each member of ${}^*\mathbf{T}$ and ${}^*\mathbf{T}'$ are subtle consequence operators.*

PROOF. Let $R = \{(x, y) \mid x, y \in \mathbf{T} \text{ and } x \leq y\}$ and $R' = \{(x, y) \mid x, y \in \mathbf{T}' \text{ and } x < y\}$. In the usual manner, it follows that R and R' are concurrent on the set \mathbf{T} and \mathbf{T}' respectively. Thus there is some $C_0 \in {}^*\mathbf{T}$ and $C'_0 \in {}^*\mathbf{T}'$ such that for each $C \in \mathbf{T}$ and $C' \in \mathbf{T}'$, ${}^*\mathbf{C} \leq C_0$ and ${}^*\mathbf{C}' < C'_0$ since ${}^*\mathcal{M}$ is an enlargement. Note that the members of ${}^*\mathbf{T}$ and ${}^*\mathbf{T}'$ are defined on the set of all internal subsets of ${}^*\mathbf{L}$. However, if there is some similarly defined $D \in \mathcal{N}$ such that C_0 or $C'_0 = {}^\sigma D$, then since ${}^\sigma D$ is only defined for * -extensions of the (standard) members of $\mathcal{P}(L)$ and each $E \in {}^*\mathbf{T}$ or ${}^*\mathbf{T}'$ is defined on the internal subsets of ${}^*\mathbf{L}$ and there are internal subsets of ${}^*\mathbf{L}$ that are not * -extensions of standard sets we would have a contradiction. Of course, each member of ${}^*\mathbf{T}$ or ${}^*\mathbf{T}'$ is a subtle consequence operator. Hence each $E \in {}^*\mathbf{T}$ or ${}^*\mathbf{T}'$ is either equal to some ${}^*\mathbf{C}$, where $C \in \mathbf{T}$ or $C \in \mathbf{T}'$ or it is a purely subtle consequence operator. Now there does not exist a $D \in \mathcal{N}$ such that $C'_0 = {}^*\mathbf{D}$ since $C'_0 \in {}^*\mathbf{T}'$ and ${}^*\mathbf{C} \neq C'_0$ for each ${}^*\mathbf{C} \in {}^\sigma\mathbf{T}'$ would yield the contradiction that ${}^*D \in {}^*\mathbf{T}' - {}^\sigma\mathbf{T}'$ but ${}^*\mathbf{D} \in {}^\sigma\mathcal{C}$. Hence C'_0 is a purely subtle consequence operator. This completes the proof.

Let $C \in \mathbf{T}'$. Since ${}^*\mathbf{C} < C'_0$, then C'_0 is “more powerful” than any $C \in \mathbf{T}'$ in the following sense. If $B \in \mathcal{P}(L)$, then for each $C \in \mathbf{T}'$ it follows that $\mathbf{C}(B) \subset {}^*(\mathbf{C}(B)) = {}^*\mathbf{C}({}^*B) \subset C'_0({}^*B)$. Recall that, for $C \in \mathcal{C}$, a set $B \in \mathcal{P}(L)$ is called a *C-deductive system* if $C(B) = B$. From this point on, all results are restricted to chains in $\langle \mathcal{C}_f, \leq \rangle$.

THEOREM 4.3. *Let $C \in \mathbf{T} \cup \mathbf{T}'$ and $B \in \mathcal{P}(L)$. Then there exists a * -finite $F_0 \in {}^*(F(B))$ such that $\mathbf{C}(B) \subset {}^*\mathbf{C}(F_0) \subset {}^*\mathbf{C}({}^*B) = {}^*(\mathbf{C}(B))$ and ${}^*\mathbf{C}(F_0) \cap \mathbf{L} = \mathbf{C}(B) = {}^*\mathbf{C}(F_0) \cap \mathbf{C}(B)$.*

PROOF. Consider the binary relation $Q = \{(x, y) \mid x \in \mathbf{C}(B), y \in F(B) \text{ and } x \in \mathbf{C}(y)\}$. By axiom (iii), the domain of Q is $\mathbf{C}(B)$. Let $(x_1, y_1), \dots, (x_n, y_n) \in Q$. By theorem 1 in [6, p. 64], (the monotone theorem) we have that $\mathbf{C}(y_1) \cup \dots \cup \mathbf{C}(y_n) \subset \mathbf{C}(y_1 \cup \dots \cup y_n)$. Since $F = y_1 \cup \dots \cup y_n \in F(B)$, then $(x_1, F), \dots, (x_n, F) \in Q$. Thus Q is concurrent on $\mathbf{C}(B)$. Hence there

exists some $F_0 \in {}^*(F(\mathbf{B}))$ such that $\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset {}^*\mathbf{C}(F_0) \subset {}^*\mathbf{C}({}^*\mathbf{B}) = {}^*(\mathbf{C}(\mathbf{B}))$. Since $\sigma\mathbf{L} = \mathbf{L}$, then ${}^*\mathbf{C}(F_0) \cap \mathbf{L} = \mathbf{C}(\mathbf{B}) = {}^*\mathbf{C}(F_0) \cap \mathbf{C}(\mathbf{B})$.

COROLLARY 4.3.1 *If $C \in \mathcal{C}_f$ and $\mathbf{B} \in \mathcal{P}(\mathbf{L})$ is a C -deductive system, then there exists a * -finite $F_0 \subset {}^*\mathbf{B}$ such that ${}^*\mathbf{C}(F_0) \cap \mathbf{L} = \mathbf{B}$.*

PROOF. Simply consider the one element chain $\mathbf{T} = \{C\}$.

COROLLARY 4.3.2. *Let $C \in \mathcal{C}_f$. There there exists a * -finite $F_1 \subset {}^*\mathbf{L}$ such that for each C -deductive system $\mathbf{B} \subset \mathbf{L}$, ${}^*\mathbf{C}(F_1) \cap \mathbf{B} = \mathbf{B}$.*

PROOF. Let $\mathbf{T} = \{C\}$ and the “ \mathbf{B} ” in theorem 4.3 equal \mathbf{L} . The result now follows in a straightforward manner.

THEOREM 4.4. *Let $\mathbf{B} \in \mathcal{P}(\mathbf{L})$.*

(i) *There exists a * -finite $F_B \in {}^*(F(\mathbf{B}))$ and a subtle consequence operator $C_B \in {}^*\mathbf{T}$ such that for all $C \in \mathbf{T}$, $\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset C_B(F_B)$.*

(ii) *There exists a * -finite $F_B \in {}^*(F(\mathbf{B}))$ and a purely subtle consequence operator $C'_B \in {}^*\mathbf{T}'$ such that for all $C \in \mathbf{T}'$, $\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset C'_B(F_B)$.*

PROOF. Consider the “binary” relation $Q = \{((x, z), (y, w)) \mid x \in \mathbf{T}, y \in \mathbf{T}, w \in F(\mathbf{B}), z \in x(w), z \in \mathcal{P}(\mathbf{L}), z \in x(w), \text{ and } x(w) \subset y(w)\}$. Let $\{((x_1, z_1), (y_1, w_1)), \dots, ((x_n, z_n), (y_n, w_n))\} \subset Q$. Notice that $F = w_1 \cup \dots \cup w_n \in F(\mathbf{B})$ and for the set $K = \{x_1, \dots, x_n\}$, let D be the largest member of K with respect to \leq . It follows that $z_i \in x_i(w_i) \subset x_i(F) \subset D(F)$ for each $i = 1, \dots, n$. Hence $\{((x_1, z_1), (D, F)), \dots, ((x_n, z_n), (D, F))\} \subset Q$ implies that Q is concurrent on its domain. Consequently, there exists some $(C_B, F_B) \in {}^*\mathbf{T} \times {}^*(F(\mathbf{B}))$ such that for each $(x, z) \in \text{domain of } Q$, $({}^*(x, z), (C_B, F_B)) \in {}^*Q$. Or, for each $(u, v) \in \sigma(\text{domain of } Q)$, $((u, v), (C_B, F_B)) \in {}^*Q$. Let arbitrary $C \in \mathbf{T}$ and $b \in C(\mathbf{B})$. Then there exists some $F' \in F(\mathbf{B})$ such that $\mathbf{b} \in \mathbf{C}(F')$. Thus $({}^*\mathbf{C}, {}^*\mathbf{b}) \in \sigma(\text{domain of } Q)$. Consequently, for each $C \in \mathbf{T}$ and $\mathbf{b} \in \mathbf{C}(\mathbf{B})$, $\mathbf{b} = {}^*\mathbf{b} \in ({}^*\mathbf{C})(F_B) \subset C_B(F_B)$. This all implies that for each $C \in \mathbf{T}$, $\sigma(\mathbf{C}(\mathbf{B})) = \mathbf{C}(\mathbf{B}) \subset C_B(F_B)$.

(ii) Change the relation Q to Q' by requiring that $x \neq y$. Replace D in the proof of (i) above with D' is greater than and not equal to the largest member of K . Such a D' exists in \mathbf{T}' from the definition of \mathbf{T}' . Continue the proof in the same manner in order to obtain C'_B and F'_B . The fact that C'_B is a purely subtle consequence operator follows in the same manner as in the proof of theorem 4.2.

COROLLARY 4.4.1 *There exists a [resp. purely] subtle consequence operator $C_L \in {}^*\mathbf{T}$ [resp. ${}^*\mathbf{T}'$] and a * -finite $F_L \in {}^*(F(\mathbf{L}))$ such that for all $C \in \mathbf{T}$ [resp. \mathbf{T}'] and each $\mathbf{B} \in \mathcal{P}(\mathbf{L})$, $\mathbf{B} \subset \mathbf{C}(\mathbf{B}) \subset C_L(F_L)$.*

PROOF. Simply let “ \mathbf{B} ” in theorem 4.4 be equal to \mathbf{L} . Then there exists a [resp. purely] subtle $C_L \in {}^*\mathbf{T}$ [resp. ${}^*\mathbf{T}'$] and $F_L \in {}^*(F(\mathbf{L}))$ such that for all $C \in \mathbf{T}$ [resp. \mathbf{T}'] $\mathbf{C}(\mathbf{L}) \subset C_L(F_L)$. If $\mathbf{B} \in \mathcal{P}(\mathbf{L})$ and $C \in \mathbf{T}$ [resp. \mathbf{T}'], then $\mathbf{B} \subset \mathbf{C}(\mathbf{B}) \subset \mathbf{C}(\mathbf{L})$. Thus for each $\mathbf{B} \in \mathcal{P}(\mathbf{L})$ and $C \in \mathbf{T}$ [resp. \mathbf{T}'] $\mathbf{B} \subset \mathbf{C}(\mathbf{B}) \subset C_L(F_L)$ and the theorem is established.

The nonstandard results in this section have important applications to mathematical philosophy. We present two such applications. Let \mathcal{F} be the symbolic alphabet for any formal language \mathbf{L} with the usual assortment of primitive symbols [10, p. 59]. We note that it is possible to mimic the construction of \mathbf{L} within \mathcal{E} itself. If this is done, then it is not necessary to consider the intuitive map θ and we may restrict our attention entirely to the sets \mathcal{E} and ${}^*\mathcal{E}$.

Let S denote the predicate consequence operator by the standard rules for predicate (proof-theory) deduction as they appear on pages 59 and 60 of reference [10]. Hence $A \in \mathcal{P}(L)$, $S(A) = \{x \mid x \in L \text{ and } A \vdash x\}$. It is not difficult to restrict the modus ponens rule of inference in such a manner that a denumerable set $T' = \{C_n \mid n \in \mathbb{N}\}$ of consequence operators defined on $\mathcal{P}(L)$ is generated with the following properties.

- (i) For each $A \in \mathcal{P}(L)$, $S(A) = \bigcup\{C_n(A) \mid n \in \mathbb{N}\}$ and $C_n \neq S$ for any $n \in \mathbb{N}$.
- (ii) For each $C \in T'$ there is a C' such that $C < C'$ [5, p.57]. Let $A \in \mathcal{P}(L)$ be any S -deductive system. The $A = S(A) = \bigcup\{C_n(A) \mid n \in \mathbb{N}\}$ yields that A is a C_n -deductive system for each $n \in \mathbb{N}$. Thus S and C_n $n \in \mathbb{N}$ are consequence operators defined on $\mathcal{P}(A)$ as well as on $\mathcal{P}(L)$.

THEOREM 4.5. *Let L be a first-order language and $A \in \mathcal{P}(L)$. Then there exists a purely subtle $C_1 \in {}^*T'$ and a * -finite $F_1 \in {}^*(F(A))$ such that for each $B \in \mathcal{P}(A)$ and each $C \in T'$*

- (i) $C(B) \subset C_1(F_1)$,
- (ii) $S(B) \subset C_1(F_1) \subset {}^*S(F_1) \subset {}^*(S(A))$.
- (iii) ${}^*S(F_1) \cap L = S(A) = C_1(F_1) \cap L$.

PROOF. The same proof as for corollary 4.4.1 yields that there is some purely subtle $C_1 \in {}^*T'$ and $F_1 \in {}^*(F(A))$ such that for each $B \in \mathcal{P}(A)$ and each $C \in T'$, $C(B) \subset C_1(F_1)$ and (i) follows. From (i), it follows that $\bigcup\{\sigma(C(B)) \mid C \in T'\} = \bigcup\{C(B) \mid C \in T'\} = S(B) = \sigma(S(B)) \subset C_1(F_1)$ and the first part of (ii) holds. By * -transfer $C_1 < {}^*S$ and C_1 and *S are defined on internal subsets of *A . Thus $C_1(F_1) \subset {}^*S(F_1) \subset {}^*S({}^*A) = {}^*(S(A))$ by the * -monotone property. This completes (ii). Since $S(A) \subset C_1(F_1) \subset {}^*S(F_1) \subset {}^*(S(A))$ from (ii), then (iii) follows and the theorem is proved.

REMARK 4.6. Of course, it is well known that there exists some $F \in {}^*(F(A))$ such that $S(A) \supset A \subset F \subset {}^*A$ and * -transfer of axiom (i) yields that ${}^*S(F) \subset {}^*S({}^*A) = {}^*(S(A))$. However, F_1 of theorem 4.5 is of a special nature in that the purely subtle C_1 applied to F_1 yields the indicated properties. Also theorem 4.5 holds for many other infinite languages and deductive processes.

Let L be a language and let M be a structure in which L can be interpreted in the usual manner. A consequence operator C is *sound* for M if whenever $A \in \mathcal{P}(L)$ has the property that $M \models A$, then $M \models C(A)$. As usual, $T(M) = \{x \mid x \in L \text{ and } M \models x\}$. Obviously, if C is sound for M , then $T(M)$ is a C -deductive system.

Corollary 4.3.1 implies that there exists * -finite $F_0 \subset {}^*(T(M))$ such that ${}^*C(F_0) \cap L = T(M)$. Notice that the fact that F_0 is * -finite implies that F_0 is * -recursive. Moreover, trivially, F_0 is a * -axiom system for ${}^*C(F_0)$, and we do not lack knowledge about the behavior of F_0 since any formal property about C or recursive sets, among others, must hold for *C or F_0 when property interpreted. If L is a first-order language, then S is sound for first-order structures. Theorem 4.5 not only yields a * -finite F_1 but a purely subtle consequence operator C_1 such that, trivially, F_1 is a * -axiom for $C_1(F_1)$ and for ${}^*S(F_1)$. In this case, we have that ${}^*S(F_1) \cap L = T(M) = C_1(F_1) \cap L$. By the use of internal and external objects, the nonstandard logics $\{{}^*C, {}^*L\}$, $\{C_1, {}^*L\}$ and $\{{}^*S, {}^*L\}$ technically by-pass a portion of Gödel's first incompleteness theorem.

By definition $b \in S(B)$, $B \in \mathcal{P}(L)$ iff there is a finite length "proof" of b from the premises B . It follows, that for each $b \in {}^*(T(M))$ there exists a * -finite length proof of b from a * -finite set of premises F_1 . If we let ${}^*\mathcal{M}$ be an enlargement with the \aleph_1 -isomorphism property, among others,

then each *-finite length proof is either externally finite or externally infinite, and all externally infinite proof lengths are of the same external cardinality [4].

References

1. A. Abian: The theory of sets and transfinite arithmetic, W. B. Saunders Co., Philadelphia and London, 1965.
2. W. Dziobiak: The lattice of strengthenings of a strongly finite consequence operator, *Studia Logica*, **40**(2) (1981), 177-193.
3. J. R. Geiser: Nonstandard logics, *J. Symbolic Logic*, **33**(1968), 236-250.
4. C. W. Henson: The isomorphism property in nonstandard analysis and its use in the theory of Banach spaces, *J. Symbolic Logic*, **39**(1974), 717-731.
5. R. A. Herrmann: The mathematics for mathematical philosophy, Monograph #130, Institute for Mathematical Philosophy Press, Annapolis, 1983. (Incorporated into "The Theory of Ultralogics," <http://xxx.arxiv.org/abs/math.GM/9903081> and 9903082.)
6. R. A. Herrmann: Mathematical Philosophy, Abstracts A. M. S. **2**(6)(1981), 527.
7. T. Tech: The axiom of choice, North-Holland and American Elsevier Publishing Co., 1973.
8. J. Los and R. Suszko: Remarks on sentential logics, *Indag. Math.*, **20**(1958), 177-183.
9. A. A. Markov: Theory of algorithms, Amer. Math. Soc. Translations, Ser. 2, **15**(1960), 1-14.
10. E. Mendelson: Introduction to mathematical logic, D. Van Nostrand, New York, 1979.
11. A. Robinson: On languages which are based on non-standard arithmetic, *Nagoya Math. J.*, **22**(1963), 83-118.
12. A. Tarski: Logic, semantics and metamathematics, Oxford University Press, Oxford 1956.
13. R. Wójcicki: Some remarks on the consequence operation in sentential logics, *Fund. Math.* **68**(1970), 269-279.

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